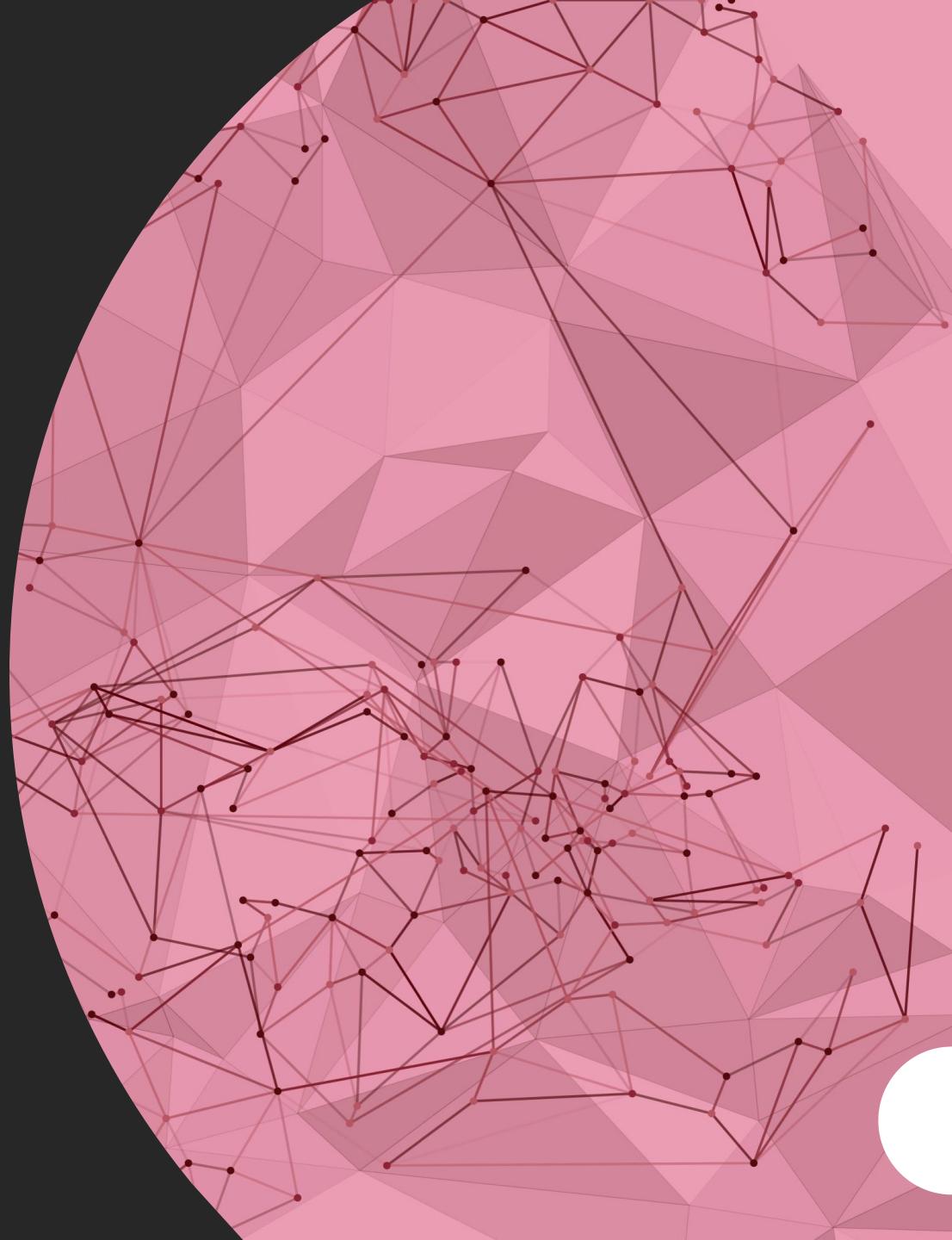


Tangent Function As A Solution Of A 3- Dimensional Functional Equation

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e.g. $f(x) = e^x$

$$f(x + y) = e^{x+y} = e^x e^y = f(x)f(y)$$

$f(x + y) = f(x)f(y)$ is a functional equation.

e.g. $g(x) = \ln(x)$

$$g(xy) = \ln(xy) = \ln(x) + \ln(y) = g(x) + g(y)$$

$g(xy) = g(x) + g(y)$ is also a functional equation.

Periodic Function: $f(x + P) = f(x)$

Even Functions: $f(x) = f(-x)$

Sine Addition Formula: $f(x + y) = f(x)g(y) + f(y)g(x)$

Applications in Mathematics

Other Applications



Euler-Lagrange Equation



Fluid Dynamics



Financial Management



Information Theory

Sum Formulas:

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

That gives $\tan(x + y) = \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)}$

Divide the top and bottom by $\cos(x)\cos(y)$

$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

$$\begin{aligned}
 \tan(x + y + z) &= \tan((x + y) + z) = \\
 &\quad \frac{\tan(x + y) + \tan(z)}{1 - \tan(x + y)\tan(z)} \\
 &= \frac{\frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)} + \tan(z)}{1 - \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}\tan(z)}
 \end{aligned}$$

Multiplying top and bottom by $1 - \tan(x)\tan(y)$

$$\tan(x + y + z) = \frac{\tan(x) + \tan(y) + \tan(z) - \tan(x)\tan(y)\tan(z)}{1 - \tan(x)\tan(y) - \tan(y)\tan(z) - \tan(z)\tan(x)}$$

We consider the following functional equation:

$$f(x + y + z) = \frac{f(x) + f(y) + f(z) - f(x)f(y)f(z)}{1 - f(x)f(y) - f(y)f(z) - f(z)f(x)}$$

has a solution $f = \tan$

Theorem:

If a differentiable function f satisfies this functional equation,
then $f(x) = \tan(cx)$ for an arbitrary real constant c .

Proof:

Let $u = x + y + z$

Then the functional equation assumes the form

$$f(u) = \frac{f(x) + f(y) + f(z) - f(x)f(y)f(z)}{1 - f(x)f(y) - f(y)f(z) - f(z)f(x)}$$

$$\begin{aligned}
& f'(u) \cdot \frac{\partial u}{\partial x} = \\
& \left[\frac{\partial}{\partial x} (f(x) + f(y) + f(z) - f(x)f(y)f(z)) (1 - f(x)f(y) - f(y)f(z) - f(z)f(x)) \right. \\
& \quad \left. - (f(x) + f(y) + f(z) - f(x)f(y)f(z)) \left(-\frac{\partial}{\partial x} (f(x)f(y) + f(y)f(z) + f(z)f(x)) \right) \right. \\
& \quad \left. \cdot \frac{1}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2} \right] \cdot \tag{3.2}
\end{aligned}$$

Now, noting that

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x + y + z) = 1,$$

$$\begin{aligned} f'(u) &= \frac{f'(x) + f'(x)f^2(y)f^2(z) + f'(x)f^2(y) + f'(x)f^2(z)}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2} \\ &= \frac{f'(x)(1 + f^2(y)f^2(z) + f^2(y) + f^2(z))}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2} \end{aligned}$$

$$f'(u) = \frac{\partial u}{\partial y} = \frac{f'(y)(1 + f^2(x)f^2(z) + f^2(x) + f^2(z))}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2}$$

$$f'(x)(1 + f^2(y))(1 + f^2(z)) = f'(y)(1 + f^2(x))(1 + f^2(z))$$

This equation gets separated as

$$\frac{f'(x)}{(1 + f^2(x))} = \frac{f'(y)}{(1 + f^2(y))}.$$

because $(1 + f^2(x)) \neq 0$ for any x , and same holds for analogous terms in y and z .

$$\frac{f'(x)}{1 + f^2(x)} = c.$$

Substituting $v = f(x)$, $\frac{dv}{1 + v^2} = c$

Integrating, $\tan^{-1}v = cx + d$

where d is an arbitrary constant.

Hence $v = \tan(cx + d)$

$f(x) = \tan(cx + d)$

Substituting $x = y = z = 0$ in the proposed functional equation:

$$f(0)[1 + (f(0))^2] = 0$$

This implies $f(0) = 0$

Using this initial condition, we have $\tan(d) = 0$
where $d = n\pi$ for an arbitrary integer n .

Since $f(x) = \tan(cx + d)$,
 $f(x) = \tan(cx)$.

This completes the proof.

References

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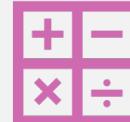
Main Takeaways and Q&A



Functional equations have implications in many fields.



We found that a tangent function is a solution to a 3-dimensional functional equation.



Methods: differential equations, multivariable calculus.