Tangent Function As A Solution Of A 3-Dimensional Functional Equation

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$$\text{e.g. } f(x) = e^x$$

$$f(x+y) = e^{x+y} = e^x e^y = f(x)f(y)$$

$f(x+y) = f(x)f(y)$ is a functional equation.

$$\text{e.g. } g(x) = \ln(x)$$

$$g(xy) = \ln(xy) = \ln(x) + \ln(y) = g(x) + g(y)$$

$g(xy) = g(x) + g(y)$ is also a functional equation.

Periodic Function: $f(x+P) = f(x)$

Even Functions: $f(x) = f(-x)$

Sine Addition Formula: $f(x+y) = f(x)g(y) + f(y)g(x)$

Applications in Mathematics

Other Applications

- Euler-Lagrange Equation
- Fluid Dynamics
- Financial Management
- Information Theory

Sum Formulas:

$$sin(x+y) = sin(x)cos(y) + cos(x)sin(y)$$

$$cos(x+y) = cos(x)cos(y) - sin(x)sin(y)$$

That gives $tan(x+y) = \frac{sin(x)cos(y) + cos(x)sin(y)}{cos(x)cos(y) - sin(x)sin(y)}$

Divide the top and bottom by $cos(x)cos(y)$

$$tan(x+y) = \frac{tax(x) + tan(y)}{1 - tan(x)tan(y)}$$

$$tan(x+y+z) = tan((x+y)+z) =$$

$$\frac{tan(x+y)+tan(z)}{1-tan(x+y)tan(z)}$$

$$= \frac{\frac{tan(x)+tan(y)}{1-tan(x)tan(y)} + tan(z)}{1 - \frac{tan(x)+tan(y)}{1-tan(x)tan(y)}tan(z)}$$

$$\text{Multiplying top and bottom by } 1 - tan(x)tan(y)$$

$$tan(x+y+z) = \frac{tan(x) + tan(y) + tan(z) - tan(x)tan(y)tan(z)}{1 - tan(x)tan(y) - tan(y)tan(z) - tan(z)tan(x)}$$

We consider the following functional equation:

$$f(x+y+z) = \frac{f(x)+f(y)+f(z)-f(x)f(y)f(z)}{1-f(x)f(y)-f(y)f(z)-f(z)f(x)}$$

has a solution $f = tan$

Theorem:

If a differentiable function f satisfies this functional equation, then $f(x) = tan(cx)$ for an arbitrary real constant c.

Proof:

Let $u = x + y + z$

Then the functional equation assumes the form

$$f(u) = \frac{f(x) + f(y) + f(z) - f(x)f(y)f(z)}{1 - f(x)f(y) - f(y)f(z) - f(z)f(x)}$$

$$
\begin{aligned}
&f'(u)\cdot\frac{\partial u}{\partial x} = \\
&[\frac{\partial}{\partial x}(f(x)+f(y)+f(z)-f(x)f(y)f(z))(1-f(x)f(y)-f(y)f(z)-f(z)f(x)) \\
&-(f(x)+f(y)+f(z)-f(x)f(y)f(z))(-\frac{\partial}{\partial x}(f(x)f(y)+f(y)f(z)+f(z)f(x))) \\
&]\cdot\frac{1}{(1-f(x)f(y)-f(y)f(z)-f(z)f(x))^2}. \qquad (3.2)
\end{aligned}
$$

Now, noting that

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x + y + z) = 1,$$

$$f'(u) = \frac{f'(x) + f'(x)f^2(y)f^2(z) + f'(x)f^2(y) + f'(x)f^2(z)}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2}$$

$$= \frac{f'(x)(1 + f^2(y)f^2(z) + f^2(y) + f^2(z))}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2}$$

$$f'(u) = \frac{\partial u}{\partial y} = \frac{f'(y)(1 + f^2(x)f^2(z) + f^2(x) + f^2(z))}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2}$$

$$f'(x)(1+f^2(y))(1+f^2(z)) = f'(y)(1+f^2(x))(1+f^2(z))$$

This equation gets separated as

$$\frac{f'(x)}{(1+f^2(x))} = \frac{f'(y)}{(1+f^2(y))}.$$

because $(1+f^2(x)) \neq 0$ for any x, and same holds for analogous terms in y and z.

$$\frac{f'(x)}{1+f^2(x)} = c$$

$$\text{Substituting } v = f(x),\ \frac{dv}{1+v^2} = c$$

$$\text{Integrating, } tan^{-1} v = cx + d$$

where d is an arbitrary constant.

$$\text{Hence } v = tan(cx + d)$$

$$f(x) = tan(cx + d)$$

Substituting $x = y = z = 0$ in the proposed functional equation:

$$f(0)[1 + (f(0))^2] = 0$$

This implies $f(0) = 0$

Using this initial condition, we have $tan(d) = 0$

where $d = n\pi$ for an arbitrary integer n.

Since $f(x) = tan(cx + d)$,

$f(x) = tan(cx)$.

This completes the proof.

References

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Main Takeaways and Q&A

Functional equations have implications in many fields.

We found that a tangent function is a solution to a 3-dimensional functional equation.

Methods: differential equations, multivariable calculus.