TANGENT FUNCTION AS A SOLUTION OF A 3-DIMENSIONAL FUNCTIONAL EQUATION

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Abstract

Taking a cue from the identity for the tangent of the sum of three angles, we form a corresponding functional equation in three variables. Then, supposing the function to be differentiable, we solve the functional equation and conclude that the solution is essentially the tangent function up to a constant multiple of the argument variable. The methodology used can be understood by a student with a background in multivariable calculus.

1 Introduction

Functional equations are equations in which the unknowns are functions, rather than a traditional variable. One of the applications of functional equations is that they can be used to characterizing the elementary functions. They have applications in mathematical physics, fluid mechanics and economics. For details we refer to Aczel [1], Aczel and Dhombres [2], Castillo et al. [3], Small [5] and Efthimiou [4]. In general, solving a functional equation assumes the continuity of the unknown function. In this short article, we will solve a functional equation involving three variables assuming the differentiability of the unknown function. This approach would be accessible to an undergraduate student with a background of multivariable calculus.

2 Problem And The Solution

Let us recall the following well known trigonometric identity for the tangent of sum of three angles

$$tan(x+y+z) = \frac{tanx + tany + tanz - (tanx)(tany)(tanz)}{1 - (tanx)(tany) - (tany)(tanz) - (tanz)(tanx)} \quad (2.1)$$

Motivated by this identity, we consider the following functional equation

$$f(x+y+z) = \frac{f(x) + f(y) + f(z) - f(x)f(y)f(z)}{1 - f(x)f(y) - f(y)f(z) - f(z)f(x)} \quad (2.2)$$

in the unknown real valued function f. Evidently, one solution is $f = tan$. So, the question arises "What is the most general solution of the aforementioned functional equation?". We provide the answer by proving the following result.

Theorem 2.1 *If a differentiable function f satisfies the functional equation (2.2), then $f(x) = tan(cx)$ for an arbitrary real constant c.*

3 Proof of the Theorem

We begin by setting $u = x + y + z$. Then the functional equation (2.2) assumes the form

$$f(u) = \frac{f(x) + f(y) + f(z) - f(x)f(y)f(z)}{1 - f(x)f(y) - f(y)f(z) - f(z)f(x)}. \tag{3.1}$$

As f is differentiable by our hypothesis, differentiating (3.1) partially with respect to x, we get

$$\begin{aligned}
&f'(u) \cdot \frac{\partial u}{\partial x} = \\
&[\frac{\partial}{\partial x}(f(x) + f(y) + f(z) - f(x)f(y)f(z))(1 - f(x)f(y) - f(y)f(z) - f(z)f(x)) \\
&-(f(x) + f(y) + f(z) - f(x)f(y)f(z))(-\frac{\partial}{\partial x}(f(x)f(y) + f(y)f(z) + f(z)f(x))) \\
&] \cdot \frac{1}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2}.
\end{aligned} \tag{3.2}$$

Now, noting that

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x + y + z) = 1, \tag{3.3}$$

we can simplify and write (3.2) as the following equation

$$\begin{aligned}
f'(u) &= \frac{f'(x) + f'(x)f^2(y)f^2(z) + f'(x)f^2(y) + f'(x)f^2(z)}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2} \\
&= \frac{f'(x)(1 + f^2(y)f^2(z) + f^2(y) + f^2(z))}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2}
\end{aligned} \tag{3.4}$$

where $f^2(x)$ stands for $(f(x))^2$. Similarly, differentiating (3.1) with respect to y provides the equation

$$f'(u) = \frac{\partial u}{\partial y} = \frac{f'(y)(1 + f^2(x)f^2(z) + f^2(x) + f^2(z))}{(1 - f(x)f(y) - f(y)f(z) - f(z)f(x))^2}. \qquad (3.5)$$

We get a similar equation by differentiating (3.1) with respect to z. Now comparing the equations (3.4) and (3.5) and noting that the denominators are identical, we find

$$f'(x)(1 + f^2(y))(1 + f^2(z) = f'(y)(1 + f^2(x))(1 + f^2(z)). \qquad (3.6)$$

This equation gets separated as

$$\frac{f'(x)}{(1 + f^2(x))} = \frac{f'(y)}{(1 + f^2(y))}. \qquad (3.7)$$

because $(1 + f^2(x)) \neq 0$ for any x, and same holds for analogous terms in y and z.

At this point, we observe that the left side of the equation (3.7) depends only on x, whereas the right side depends only on y. This is possible only when each side is equal to a constant, say c. Thus we obtain the following differential equation in $f(x)$:

$$\frac{f'(x)}{1 + f^2(x)} = c, \qquad (3.8)$$

Substitution $v = f(x)$ turns the above equation into $\frac{dv}{1+v^2} = c$. Integrating it we get $tan^{-1}v = cx + d$ where d is an arbitrary constant. Hence $v = tan(cx + d)$, i.e.

$$f(x) = \tan(cx + d), \qquad (3.9)$$

Next, substituting $x = y = z = 0$ in (2.2) shows that

$$f(0)[1 + (f(0))^2] = 0. \qquad (3.10)$$

This readily implies that $f(0) = 0$. The use of this initial condition in (3.9) gives $tan(d) = 0$. Hence $d = n\pi$ for an arbitrary integer n. This, in conjunction with (3.9), yields that $f(x) = tan(cx)$. This completes the proof.

References

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4 Student Biography

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